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Characterization of f-Vectors of Families of Convex Sets in $\mathbb{R}^d$ Part II: Sufficiency of Eckhoff's Conditions

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Let $\mathscr{K}=\{K_1,..., K_n\}$ be a family of n convex sets in $\mathbb{R}^d$. For $0\leqslant i<n$ denote by f_i the number of subfamilies of $\mathscr{K}$ of size $i+1$ with non-empty intersection. The vector $f(\mathscr{K})=(f_0, f_1,...)$ is called the f-vector of $\mathscr{K}$. In 1973, Eckhoff proposed a characterization of the set of f-vectors of finite families of convex sets in $\mathbb{R}^d$ by a system of inequalities. In part I we proved the necessity of Eckhoff's inequalities and here we prove their sufficiency.

1. INTRODUCTION

Let $\mathscr{K}=\{K_1, K_2,..., K_n\}$ be a family of convex sets in $\mathbb{R}^d$. The nerve of K, $N(\mathscr{K})$ is a simplicial complex on the vertex set $N=\{1, 2,..., n\}$ defined by

$$N(\mathscr{K})=\left\{S\subset N: \bigcap_{i\in S} K_i\neq\varnothing\right\}. \tag{1.1}$$

A simplicial complex C is d-representable if $C=N(\mathscr{K})$ for a finite family $\mathscr{K}$ of convex sets in $\mathbb{R}^d$. Let $f(C)=(f_0, f_1,...)$ be the f-vector of C (thus, $f_i(C)$ is the number of i-dimensional faces of C). Then f_i is just the number of intersecting subfamilies of $\mathscr{K}$ of size $i+1$.

In 1973, Eckhoff [Ec1, Ec2] conjectured that f-vectors of d-representable complexes are characterized by a system of inequalities (see (1.5) below). In Part I ([Ka1]) we proved the necessity of Eckhoff's conditions, and here we prove their sufficiency by constructing appropriate families of convex sets. An independent proof of the sufficiency part of the conjecture was found by Eckhoff but was not published.

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Let $d>0$ be a *fixed* integer. For an ultimately vanishing sequence $f=(f_0, f_1,...)$ of integers define another such sequence $h=h[f]=(h_0, h_1,...)$ by:

$$\begin{aligned} h_k &= f_k & k&=0, 1,..., d-1, \\ &= \sum_{j\geq 0} (-1)^j \binom{k+j-d}{j} f_{k+j} & k&=d, d+1,..., \end{aligned} \tag{1.2}$$

When $f=f(C)$ for some simplicial complex C then $h[f]$ is called the h-vector of C and is denoted by $h(C)$.

For positive integers l and k, l can be written uniquely in the form:

$$l=\binom{l_k}{k}+\binom{l_{k-1}}{k-1}+\cdots+\binom{l_j}{j}, \tag{1.3}$$

where $l_k > l_{k-1} > \cdots > l_j \geqslant j \geqslant 1$. Given this reresentation define

$$l^{(k)}=\binom{l_k}{k-1}+\binom{l_{k-1}}{k-2}+\cdots+\binom{l_j}{j-1}. \tag{1.4}$$

Define also $0^{(k)}=0$. We are now ready to formulate Eckhoff's conjecture:

THEOREM 1.1. *$h=(h_0, h_1,...)$ is the h-vector of a d-representable complex iff the following inequalities hold*:

$$\begin{aligned} &\text{(i)} & h_k &\geqslant 0, & k&=0, 1,..., \\ &\text{(ii)} & h_k^{(k+1)} &\leqslant h_{k-1}, & k&=1,..., d-1, \\ &\text{(iii)} & h_k^{(d)} &\leqslant h_{k-1}-h_k, & k&=d, d+1,.... \end{aligned} \tag{1.5}$$

Preliminary concepts will be developed in Sections 2, 3, 4, and 6. In Section 5 we will construct for a given h-vector satisfying inequalities (1.5) a simplicial complex $C=C(h)$ such that $h(C)=h$. The construction of a family $\mathscr{K}$ of convex sets in $\mathbb{R}^d$ such that $N(\mathscr{K})=C$ is given in Sections 7–8. The starting point of the construction is an arrangement of hyperplanes in $\mathbb{R}^d$ which is dual, in a sense, to the set of vertices of a cyclic polytope. In Section 7 we construct a family $\mathscr{K}(t_1,..., t_n)$ of convex sets in $\mathbb{R}^d$, which depends on real numbers $t_1,..., t_n$. We show there that $C(h)$ is always contained in $N(\mathscr{K}(t_1,..., t_n))$. In Section 8 we finally prove the existence of real numbers $\bar{t}_1,..., \bar{t}_n$ for which $C(h)=N(\mathscr{K}(\bar{t}_1,..., \bar{t}_n))$. The proof is done using appropriate calculations in the ordered field of rational functions in n variables over $\mathbb{R}$.

The characterization of f-vectors of d-representable complexes is a small fragment of the study of combinatorial properties of d-representable complexes, and the more general area of "Helly-type theorems." For a survey of the developments in this area until 1963 see [DGK]. It is our hope that the methods developed in both parts of this paper will be useful in studying further problems in this area.

The following notations are used in the paper: For a natural number m, $[m]=\{1, 2,..., m\}$. For $j>i\geqslant 0$, $[i, j]=\{i, i+1,..., j\}$. For a set T and a nonnegative integer r, $T^{(r)}=\{S\subset T: |S|=r\}$, and $T^{(\leqslant r)}=\{S\subset T: |S|\leqslant r\}$. For a simplicial complex C and a nonnegative integer k,

$$\operatorname{skel}_k C=\{S\in C: \dim S\leqslant k\}(=\{S\in C: |S|\leqslant k+1\}).$$

$$C_k=\{S\in C: \dim S=k\}.$$

$T=\{i_1<i_2<\cdots<i_j\}$ will mean $T=\{i_1, i_2,..., i_j\}$ and $i_1<i_2<\cdots<i_j$. $\mathbb{N}$ denotes the set of natural numbers. For a complex C and a face $S\in C$, the quotient complex C/S (known also as the link of S in C) is defined by

$$C/S=\{T\backslash S: T\in C, T\supset S\}.$$

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2. d-Collapsible Simplicial Complexes

The notion of a d-collapsible simplicial complex played an important role in the proof of the necessity of Eckhoff's conditions. There we used Wegner's fundamental theorem which asserts that every d-representable complex is d-collapsible.

In the proof of the sufficiency, basic properties of d-collapsible complexes, and those of a more restricted family of complexes, the family of d-canonical complexes, is useful in proving the basic properties of the complex $C(h)$ constructed in Section 5. In this section the definition of d-collapsible complexes is given and some easy results concerning them are quoted. The family of d-canonical complexes is introduced in the next section.

Definition 2.1. A face S of a simplicial complex C is *free* if S is included in a unique maximal face of C.

Definition 2.2. Let C be a simplicial complex. A special elementary d-collapse is one of the following two operations:

(A) Removal of a maximal face of dimension $<d$.

(B) Removal of a free face S of dimension $d-1$ and all the faces that include S.

We shall say that a special elementary collapse is of type $(\mathrm{A})_r$ $(r<d)$ $[\mathrm{B}_r$ $(r\geqslant d)]$, if it is of type (A) [(B)] and the maximal removed face is r-dimensional.

DEFINITION 2.3. A finite simplicial complex C is called *d-collapsible* if C can be reduced to the void complex by a sequence of special elementary d-collapses.

Remark. Definition 2.3 is slightly different from the definition in Part I, however, they are equivalent by [I, Lemma 2.4].

We shall need the following easy lemma.

LEMMA 2.4. (i) *If $C \to C'$ is a special elementary d-collapse of type $(\mathrm{A})_e$ $(e<d)$ then $h_e(C)=h_e(C')+1$, and $h_k(C)=h_k(C')$ for all $k \neq e$.*

(ii) *If $C \to C'$ is a special elementary d-collapse of type $(\mathrm{B})_r$ $(r\geqslant d)$, then*

$$h_k(C)=h_k(C')+1 \qquad \text{for} \quad d-1 \leqslant k \leqslant r$$

and

$$h_k(C)=h_k(C') \qquad \textit{otherwise.}$$

3. d-CANONICAL COMPLEXES

Let $d>0$ be a fixed integer.

DEFINITION 3.1. A simplicial complex C on the vertex set $[n]$ is *d-canonical* if for every $j \geqslant 0$ and for every maximal face M of dimension $d+j$, $M \supset [j+1]$.

Obviously every complex C such that $\dim C<d$ is d-canonical. For a simplicial complex C define

$$C_k^0=\{S \in C\colon \dim S=k,\ S \supset [k-d+1]\}. \tag{3.1}$$

(Thus, for $k<d$, $C_k^0=C_k$, and C is d-canonical iff every maximal face of C belongs to $\bigcup_{k\geqslant 0} C_k^0$).

PROPOSITION 3.2. *If C is d-canonical then C is d-collapsible. Moreover C can be reduced to the void complex by $|C_k^0|-|C_{k+1}^0|$ special elementary d-*

collapse steps of type $(\mathrm{B})_k$ *for* $k \geqslant d$, *and* $|C_k^0|$ *special elementary* d*-collapse steps of type* $(\mathrm{A})_k$ *for* $k < d$.

Proof. Let C be a d-canonical complex. Put $t = \dim C$, and let $T \in C_t$. If $t < d$ define $C' = C \backslash \{T\}$. Obviously $C \to C'$ is a special d-collapse step of type $(\mathrm{A})_t$. Obviously C' is d-canonical, $|C_k'^0| = |C_k^0|$ for $k < t$ and $|C_t'^0| = |C_t^0| - 1$.

If $t \geqslant d$, denote $t = d + r$. T is maximal face in C and therefore $T \in C_{d+r}^0$, i.e., $T = [r+1] \cup F$, where $F \in [r+2, n]^{(d)}$. F is a free face of C, indeed if S is any maximal face containing F, $\dim S = d + j$, then $S \in C_{d+j}^0$, and therefore $S = [j+1] \cup F$, thus $S \subset T$ and therefore $S = T$.

Define

$$C' = C \backslash \{S: S \supset F\}.$$

$C \to C'$ is a special d-collpase step of type $(\mathrm{B})_{d+r}$. We prove now the following two assertions:

(a) C' is d-canonical;

(b) $|C_k'^0| = |C_k^0|$ for $k < d - 1$ and $|C_k'^0| = |C_k^0| - 1$ for $d - 1 \leqslant k \leqslant d + r$.

Proof of (a). Let M be a maximal face of C' of dimension $d + j$; we have to show that $M \supset [j+1]$. If M is maximal in C then $M \supset [j+1]$ (C is d-canonical). Otherwise, $M \subset T$, and since $M \in C'$, $M \not\supset F$. Clearly $M \subset (M \cap F) \cup [r+1] \in C$ and therefore $M \subset (M \cap F) \cup [r+1]$, and thus $M \supset [r+1] \supseteq [j+1]$. ∎

Proof of (b). The assertion is trivial for $k < d$. For $k = d + j$, $C_k^0 \backslash C_k'^0 = \{[j+1] \cup F\}$, and therefore $|C_k^0| = |C_k'^0| + 1$. Proposition 3.2 follows clearly by induction on C. ∎

LEMMA 3.3. $h_k(C) = |C_k^0|$.

Proof. Immediate from Proposition 3.2 and Lemma 2.4. ∎

4. PROPERTIES OF THE LEXICOGRAPHIC ORDER

Let $P_f(\mathbb{N})$ denote the set of finite subsets of $\mathbb{N}$. The lexicographic order relation $<$ on $P_f(\mathbb{N})$ is defined as follows:

DEFINITION 4.1. For $S, T \in P_f(\mathbb{N})$, $S < T$ iff $\max(S \triangle T) \in T$.

In this section we discuss some basic properties of the lexicographic order.

DEFINITION 4.2. Let $E \subset \mathbb{N}^{(k)}$ define

$$E^{\langle k|k-1\rangle} = \{S \backslash \{i\} : i \in S,\ S \in E\}.$$

We shall need the following Lemma (see, e.g., [GK]):

LEMMA 4.3. *Let E be an initial set of $\mathbb{N}^{(k)}$ w.r.t. the lexicographic order $<$, then $E^{\langle k|k-1\rangle}$ is an initial set w.r.t. $<$ and*

$$|E^{\langle k|k-1\rangle}| = |E|^{(k)}.$$

(The operation $n \to n^{(k)}$ was defined in Sect. 1.) Let $f = (f_0, f_1, ...)$ be a vector of non-negative integers which satisfies the Kruskal–Katona conditions, i.e., for all $k \geqslant 1$:

$$f_k^{(k+1)} \leqslant f_{k-1}.$$

Lemma 4.3 gives a construction of a simplicial complex C such that $f(C) = f$. For $k \geqslant 0$ let C_k be an initial set of $\mathbb{N}^{(k+1)}$ of size f_k w.r.t. the lexicographic order, and let $C_{-1} = \{\phi\}$. The required complex is defined by $C = \bigcup_{k \leqslant -1} C_k$.

DEFINITION 4.4. For $R \in \mathbb{N}^{(k)}$ put $t = \min(\mathbb{N} \backslash R)$ and $R^+ = R \cup \{t\} \in \mathbb{N}^{(k+1)}$.

It is easy to see that R^+ is the first set w.r.t. $<$ which ontains R. (In fact, R^+ is the unique minimal set that contains R w.r.t. the partial order $\lhd$ on $\mathbb{N}^{(k+1)}$ (see below).)

LEMMA 4.5. *If $R, S \subset \mathbb{N}$, $|R| < |S| < \infty$, and $R < S$ then $R^+ \leqslant S$.*

Proof. Put $s = \max S \triangle R = \max S \backslash R$, $r = \min(\mathbb{N} \backslash R)$. Clearly $s \geqslant r$. If $s > r$ then $s > x$ for each $x \in R^+ \backslash S$, and $S > R^+$. If $s = r$, then $S \backslash R = \{r\} = \{s\}$, and $S \subset R^+$, but $|S| > |R|$ and therefore $S = R^+$. ∎

LEMMA 4.6. *If $R, S \in \mathbb{N}^{(k)}$, $R < S$ then $R^+ \leqslant S^+$. (And therefore, $R^+ \leqslant S'$ for every $S' \in \mathbb{N}^{(k+1)}$, $S' \supset S$.)*

Proof. $R < S < S^+$, $|R| < |S^+|$, and by Lemma 4.5, $R^+ \leqslant S^+$. ∎

Remark. The order relation $<$ discussed in this paper is different from the opposite lexicographic order $\prec$ which was used in Part I. ($S \prec T$ iff $\min(S \triangle T) \in S$). On $\mathbb{N}^{(k)}$ these order relations are refinements of the partial order $\lhd$ defined as follows: For $S = \{i_1 < \cdots < i_k\}$, $T = \{j_1 < \cdots < j_k\}$, $S \lhd T$ iff $i_\rho \leqslant j_\rho$ for every $i \leqslant \rho \leqslant k$ and $S \neq T$.

5. Construction of the d-Collapsible Complex

Let d be a fixed positive integer. Let $f = (f_0, f_1, \ldots, f_t)$ be a vector of non-negative integers, and let $h = h(f) = (h_0, h_1, \ldots, h_t)$ be the h-vector that corresponds to f defined by Formula (1.2). We assume that h satisfies Eckhoff's conditions:

$$\begin{array}{lll} \text{(i)} & h_k \geqslant 0, & k \geqslant 0, \\ \text{(ii)} & h_k^{(k+1)} \leqslant h_{k-1}, & k \leqslant d-1, \\ \text{(iii)} & h_k^{(d)} \leqslant h_{k-1} - h_k, & k \geqslant d. \end{array} \tag{5.1}$$

For all $k \geqslant -1$ defined a subset N_k of $\mathbb{N}^{(k+1)}$ as follows:

$$\begin{aligned} N_k &= \mathbb{N}^{(k+1)} & & k < d, \\ &= \{S \in \mathbb{N}^{(k+1)} : S \supset [k-d+1]\} & & k \geqslant d. \end{aligned} \tag{5.2}$$

N_k is ordered by the lexicographic order $<$ which was defined in Section 4 ($S < T$ iff $\max(S \triangle T) \in T$).

Construction of C. For $0 \leqslant k \leqslant t$ let C_k^0 be an initial set in N_k of size h_k. Put $E = \bigcup_{k=0}^t C_k^0$. C is defined as the simplicial complex spanned by E, i.e., $S \in C$ if there exists $T \in E$ such that $S \subset T$.

For $0 \leqslant k \leqslant t$ let C_k^1 be the set of all k-faces of C which are not maximal in C. (I.e., $S \in C_k^1$ if there exists $j > k$ and $T \in C_j^0$ such that $T \supset S$.)

Theorem 5.1.

$$C_k \cap N_k = C_k^0. \tag{5.3}$$

Proof. It is clear that $C_k^0 \subset C_k \cap N_k$. It remains to prove the other inclusion. We will prove it by decreasing induction on k. For $k = t$ the assertion holds trivially. The induction step $k+1 \to k$ will be separated into two parts: $k \geqslant d-1$ and $k < d-1$.

A. Let $t > k \geqslant d-1$. We assume that (5.3) holds for $k+1$ and prove it for k. Let $S \in C_k \cap N_k$. We have to show that $S \in C_k^0$. If S is maximal the assertion follows from the definition of C. Assume that S is not maximal.

Claim 1. There exists a set T, $T \in C_{k+1}^0$, such that $S \subseteq T$.

Proof. Since S is not maximal, there are sets T', L', and an integer i such that $|L'| = i > 0$, $S = T' \backslash L'$ and $T' \in C_{k+i}^0$. Notice that $[k-d+2] \subseteq [k-d+i+1] \subseteq T'$. Let L be a subset of size $i-1$ of $L' \backslash \{k-d+2\}$. Define $T = T' \backslash L$. Then $|T| = k+2$, $T \supset [k-d+2]$, and therefore

$T \in C_{k+1} \cap N_{k+1}$. By the induction hypothesis $T \in C^0_{k+1}$ and thus S can be written in the form $T \backslash \{l\}$, where $T \in C^0_{k+1}$ and $l \in T$.

We return to the proof of the theorem. Define

$$D_1 = \{S \in C^1_k : S \supset [k+2-d]\}$$
$$D_2 = \{T \backslash \{k+2-d\} : T \in C^0_{k+1}\}.$$

It is clear that D_1 and D_2 are disjoint. From the claim it follows that $D_1 \cup D_2 = C^1_k \cap N_k$. Define

$$E_0 = C^0_{k+1}/[k+2-d] \qquad (= \{T \backslash [k+2-d] : T \in C^0_{k+1}\})$$
$$E_1 = D_1/[k+2-d] \qquad (= \{S \backslash [k+2-d] : S \in D_1\}).$$

It is clear that E_0 is an initial set of $(\mathbb{N} \backslash [k+2-d])^{(d)}$ and $E_1 = E_0^{\langle d|d-1\rangle}$. From Lemma 4.3 it follows that E_1 is an initial set of $(\mathbb{N} \backslash [k+2-d])^{(d-1)}$ and $|E_1| = |E_0|^{(d)} = h^{(d)}_{k+1}$. Now we compute $|C^1_k \cap N_k|$:

$$|D_1| = |E_1| = h^{(d)}_{k+1}$$
$$|D_2| = |C^0_{k+1}| = h_{k+1}.$$

Therefore

$$|C^1_k \cap N_k| = h^{(d)}_{k+1} + h_{k+1}.$$

From inequality (iii) it follows that

$$|C^1_k \cap N_k| = h^{(d)}_{k+1} + h_{k+1} \leqslant h_k = |C^0_k|.$$

We know that C^0_k is an initial set of N_k. The inclusion $C^1_k \cap N_k \subset C^0_k$ follows from the last inequality and the following claim:

Claim 2. $C^1_k \cap N_k$ is an initial set in N_k.

Proof. Let $S \in C^1_k \cap N_k$, $R \in N_k$, and $R < S$. We have to show that $R \in C^1_k$. From Claim 1 it follows that $S = T \backslash \{l\}$, where $T \in C^0_{k+1}$. From Lemma 4.6 it follows that $R^+ \leqslant T$, where $R^+ = R \cup \{\min(\mathbb{N} \backslash R)\}$. Since $R \supset [k+1-d]$, it follows that $T^+ \supset [k+2-d]$, and thus $R^+ \in N_{k+1}$. Since C^0_{k+1} is an initial set of N_{k+1} and $T \in C^0_{k+1}$, we have $R^+ \in C^0_{k+1}$ and therefore $R \in C^1_k$.

B. $0 \leqslant k < d-1$. Since $k+1 < d$, we have $N_{k+1} = \mathbb{N}^{(k+2)}$ and by the induction hypothesis,

$$C^0_{k+1} = C_{k+1} \cap N_{k+1} = C_{k+1}.$$

This implies

$$C_k^1 = C_{k+1}^{0\langle k+2|k+1\rangle}.$$

Since C_{k+1}^0 is an initial set of $\mathbb{N}^{(k+2)}$, C_k^1 is an initial set of $\mathbb{N}^{(k+1)}$ and

$$|C_k^1| = |C_{k+1}^0|^{(k+2)} = h_{k+1}^{(k+2)}$$

(see Lemma 4.3). But by (5.1)(ii), $h_{k+1}^{(k+2)} \leqslant h_k$ and therefore $C_k^0 \supset C_k^1$. This completes the proof of Theorem 5.1. ▮

Remark. It is possible to define $C(h)$ in the same way it was defined above for any vector h of non-negative integers. It can be proved similarly to the proof of Theorem 5.1., that $C_k \cap N_k = C_k^0$ only if h satisfies inequalities (5.1)(ii), (iii).

LEMMA 5.2. *$C = C(h)$ is a d-canonical simplicial complex.*

Proof. Immediate from the construction of C. ▮

As an immediate corollary of Theorem 5.1, Lemma 5.2, and Lemma 3.3, we obtain

LEMMA 5.3. *$h(C) = h$.*

We will specify now some properties of $C(h)$ which we shall need in the next sections. (Remember: C has n vertices.)

LEMMA 5.4. (i) *If $S, T \subset [n]$, $|T| \leqslant |S| \leqslant d$, $S \in C$, and $T \notin C$ then $T > S$.*

(ii) *For $n \geqslant i \geqslant 1$ define*

$$\mathcal{S}_i = \{S \in [i+1, n]^{(d)} : S \cup \{i\} \in C\}.$$

Then,

A. *$\mathcal{S}_i$ is an initial set w.r.t. $<$.*

B. *$\mathcal{S}_1 \supset \mathcal{S}_2 \supset \cdots \supset \mathcal{S}_{r+2} = \varnothing$, where* $\dim C = d + r$.

(iii) *If $R \subset [n]$, $|R| \geqslant d+1$, and $R \notin C$ then there exists $T \subset R$, $|T| = d+1$, and $T \notin C$.*

Proof. (i) If $|T| = |S|$ the assertion follows immediately from the construction of C. If $|T| < |S|$ it follows by repeated application of Lemma 4.5.

(ii) A and (ii)B follow easily from the construction of C.

(iii) T is the set of the last $d+1$ elements in R. ▮

6. The Ordered Polynomial Ring

Let $R = \mathbb{R}[t_1,..., t_n]$ be the ring of real polynomials in n variables $t_1,..., t_n$. A linear order is defined on R as follows: For two monomials $L = t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_n^{\alpha_n}$, $M = t_1^{\beta_1} t_2^{\beta_2} \cdots t_n^{\beta_n}$, define $L < M$ if there exists l, $n \geqslant l \geqslant 1$, such that $\beta_i = \alpha_i$ for every $i > l$ and $\beta_l > \alpha_l$. For a polynomial P, the *significant monomial* of P is the largest monomial with nonzero coefficient. $P > 0$ if the coefficient of the significant monomial of P is positive. $P > Q$ iff $P - Q > 0$. R is an ordered ring. (In fact, R is ordered by a repeated application of the standard way to order $M[x]$ for an ordered ring M.) The order of R can be extended to the field of rational functions $F = \mathbb{R}(t_1,..., t_n)$ by the following standard way: $P/Q > 0$, iff $PQ > 0$.

For $R_1, R_2 \in F$ define $R_2 \gg R_1$ if for any real number $\alpha > 0$, $R_2 > \alpha R_1$. In particular if $P, Q \in R$ and $Q > 0$ then $P \gg Q$ iff $P > 0$ and the significant monomial in P is greater than the significant monomial in Q. If $R_2 \gg R_1$ we will write $R_1 = o(R_2)$. If there exists a real $\alpha > 0$ such that $R_1 < \alpha R_2$ we will write $R_1 = O(R_2)$. $R_1 = R_2 +\ o$ will mean $R_1 = R_2 + o(R_2)$.

We shall freely use some obvious rules for operating with the symbols O and o, such as: For R_1, R_2, S_1, $S_2 \geqslant 0$, $R_1 = O(R_2)$, $S_1 = O(S_2)$ implies $R_1 S_1 = O(R_2 S_2)$; $R_1 = O(R_2)$, $S_1 = o(S_2)$ implies $R_1 S_1 = o(R_2 S_2)$; $R_1 = O(R_2)$, $S_1 = O(R_2)$ implies $R_1 + S_1 = O(R_2)$; etc.

For $\phi \neq S \subset [n]$, define $\pi(S) = \prod_{i \in S} t_i$. Put $\pi(\phi) = 1$.

We shall need the following trivial lemma:

Lemma 6.1. *If $S < T$ (Definition 4.1), then $\pi(S) < \pi(T)$.*

Definition 6.2. Let φ be a proposition in the first order theory of F. We say that φ holds for "sufficiently large" $(t_1,..., t_n)$ if there exist a constant η_1 and functions

$$\eta_2 = \eta_2(t_1), \qquad \eta_3 = \eta_3(t_1, t_2),..., \eta_n = \eta_n(t_1,..., t_{n-1})$$

such that φ holds for all $(t_1,..., t_n)$ such that

$$t_1 > \eta_1, \qquad t_2 > \eta_2(t_1),..., t_n > \eta_n(t_1,..., t_{n-1}).$$

Lemma 6.3. *If $\varphi_1, \varphi_2,..., \varphi_k$ hold for "sufficiently large" $(t_1,... t_n)$ then so does $\varphi_1 \wedge \cdots \wedge \varphi_k$.*

The proof is clear.

Lemma 6.4. *If $P \in F$, $P > 0$ then for $(t_1,..., t_n)$ sufficiently large, $P(t_1,..., t_n) > 0$.*

The proof is easy by induction on n. (See, e.g., [BL, Sect. 2].)

7. CONSTRUCTION OF THE FAMILIES OF CONVEX SETS

Let $h=(h_0, h_1,...)$ be a vector of nonnegative integers which satisfies inequalities (1.5), and let $C=C(h)$ be the simplicial complex that was defined in Section 5. As before $h_0=f_0(C)=n$. In this section and in the next one we construct a family $\mathscr{K}$ of convex sets in $\mathbb{R}^d$ such that $N(\mathscr{K})=C$, and thus complete the proof of the sufficiency of Eckhoff's conjecture.

The construction is done in three steps. In the first step we construct a family $\mathscr{M}=\{M_1,..., M_n\}$ of hyperplanes in general position in $\mathbb{R}^d$. This family is dual in a sense, to the set of vertices of a cyclic polytope. In the second step we define a family $\mathscr{L}=\{L_1,..., L_n\}$ such that each L_i is a "strip" in $\mathbb{R}^d$ that contains M_i. We will show for $S\subset[n]$ and $|S|\geqslant d$ that $S\in C\Leftrightarrow S\in N(\mathscr{L})$, but the $(d-1)$-skeleton of $N(\mathscr{L})$ is complete. In the third step we will find an appropriate polyhedral set K_0 such that $\mathscr{K}=\{K_0\cap L_1,..., K_0\cap L_n\}$ is the required family.

The families $\mathscr{M}$, $\mathscr{L}$, and $\mathscr{K}$, described above depend on variables $t_1,..., t_n$ over $\mathbb{R}$. The properties described above are proved for $(t_1,..., t_n)$ "sufficiently large" by appropriate calculations in the ordered field $F=\mathbb{R}(t_1,..., t_n)$.

In this section we define the families $\mathscr{M}$, $\mathscr{L}$, and $\mathscr{K}$ (depending on $t_1,..., t_n$) and prove that for any real numbers $t_1,..., t_n$, $N(\mathscr{K})\supset C$. In the next section we prove that for "sufficiently large" $(t_1,..., t_n)$, $N(\mathscr{K})=C$ and thus complete the proof of Theorem 1.1.

For $k\geqslant 0$ we identify $\mathbb{R}^k$ with the space of polynomials of degree $<k$:

$$(a_0, a_1,..., a_{k-1})\leftrightarrow a_0+a_1x+\cdots+a_{k-1}x^{k-1}.$$

Let $\mathbb{R}_0^k$ be the hyperplane of monic polynomials, i.e.,

$$\mathbb{R}_0^k=\{p\in\mathbb{R}^k: p=a_0+a_1x+\cdots+a_{k-2}x^{k-2}+x^{k-1}\}.$$

As usual for $p, q\in\mathbb{R}^k$, $p=\sum_{i=0}^{k-1}a_ix^i$, $q=\sum_{i=0}^{k-1}b_ix^i$, define $\langle p, q\rangle=\sum_{i=0}^{k-1}a_ib_i$. For a real number t define a hyperplane $M(t)$ in $\mathbb{R}^d$ by

$$M(t)=\{p\in\mathbb{R}^d: p(t)+t^d=0\}.$$

LEMMA 7.1. *Let $s_1, s_2,..., s_k$ be different real numbers. Then for $k\leqslant d$*

$$\bigcap_{i=1}^k M(s_i)=\left\{q(x)\prod_{i=1}^k(x-s_i)-x^d: q(x)\in\mathbb{R}_0^{d-k+1}\right\},$$

and for $k>d$

$$M(s_1)\cap\cdots\cap M(s_k)=\varnothing.$$

Proof. Let $k \leqslant d$. Then $p \in \bigcap_{i=1}^{k} M(s_i)$ iff $p(s_i) + s_i^d = 0$ for all $1 \leqslant i \leqslant k$, iff $p(x) + x^d = \prod_{i=1}^{k} (x - s_i) \cdot q(x)$, where $q(x) \in \mathbb{R}_0^{d-k+1}$.

The proof for $k > d$ is the same. ▮

We shall need a few more definitions and notations. For real numbers $s_1, s_2, \dots, s_k$, $V_k(s, \dots, s_k)$ denotes the Vandermonde matrix $(s_i^{j-1})_{1 \leqslant i \leqslant k, 1 \leqslant j \leqslant k}$, and

$$\Delta_k(s_1, \dots, s_k) = \prod_{1 \leqslant i < j \leqslant k} (s_j - s_i) = \det V_k(s_1, \dots, s_k).$$

Let $0 < t_1 < t_2 < \cdots < t_n$ be real, for $1 \leqslant i \leqslant n$ define an affine functional H_i:

$$H_i(p) = p(-t_i) + (-t_i)^d,$$

and a hyperplane M_i:

$$M_i = M(-t_i) = \{ p \in \mathbb{R}^d \colon H_i(p) = 0 \}.$$

Define also

$$M_i^+ = \{ p \colon H_i(p) \geqslant 0 \}, \qquad M_i^- = \{ p \colon H_i(p) \leqslant 0 \}.$$

The family $\mathscr{M} = \{ M_1, \dots, M_n \}$ is the starting point for the construction. For $S \subset [n]$ define $M(S) = \bigcap_{i \in S} M_i$. From Lemma 7.1 we immediately obtain that for $|S| > d$, $M(S) = \varnothing$, and for $|S| = k \leqslant d$,

$$M(S) = \left\{ q \cdot \prod_{v \in S} (x + t_v) - x^d \colon q \in \mathbb{R}_0^{d-k+1} \right\}.$$

In particular for $|S| = d$,

$$M(S) = \left\{ \prod_{v \in S} (x + t_v) - x^d \right\}.$$

Denote by $M^0(S)$ the linear subspace parallel to $M(S)$, then

$$M^0(S) = \left\{ q \cdot \prod_{v \in S} (x + t_v) \colon q \in \mathbb{R}^{d \;\; k} \right\}.$$

We will define now the families $\mathscr{L}$ and $\mathscr{K}$ as described in the beginning of this section. For $S \in [n]^{(k)}$, $k \leqslant d$, we define a point $m(S) \in M(S)$ as follows:

$$m(S) = x^{d-k} \prod_{v \in S} (x + t_v) - x^d.$$

When $|S| = d$, $m(S)$ is the unique point in $M(S)$. Define also

$$H_i(S) = H_i(m(S))(=(m(S) + x^d)(-t_i)).$$

Consider now the complex $C = C(h)$ that was defined in Section 5 and recall that

$$\mathscr{S}_i = \{S \in [i+1, n]^{(d)} : S \cup \{i\} \in C\}.$$

Define $\mathscr{L} = \{L_1, ..., L_n\}$, where

$$L_i = \overline{\text{conv}}(M_i \cup \{m(S) : S \in \mathscr{S}_i\}) \qquad 1 \leqslant i \leqslant d$$

($\overline{\text{conv}}\, X$ is the closed convex hull of X). Define

$$K_0 = \text{conv}\{m(S) : S \in \text{skel}_{d-1} C\}.$$

For $n \geqslant i \geqslant 1$ define $K_i = L_i \cap K_0$ and $\mathscr{K} = \{K_1, K_2, ..., K_n\}$. The families $\mathscr{K}$ and $\mathscr{L}$ defined above depend on $t_1, ..., t_n$. For $S \subset [n]$ define

$$K(S) = \bigcap_{i \in S} K_i, \qquad L(S) = \bigcap_{i \in S} L_i.$$

LEMMA 7.2. *$N(\mathscr{K}) \supset C$, i.e., if $S \in C$ then $K(S) \neq \varnothing$.*

Proof. If $S \in \text{skel}_{d-1} C$ then $m(S) \in M(S) \subset L(S)$ and $m(S) \in K_0$. Therefore $m(S) \in K_0 \cap L(S) = K(S)$, i.e., $K(S) \neq \varnothing$. Let $S \in C$, $|S| > d$. Let R be the set of the last d elements of S. We will show that $m(R) \in K(S)$. Clearly, $m(R) \in K_0$. It remains to show that $m(R) \in L_i$ for every $i \in S$. If $i \in R$ then $m(R) \in M_i \subset L_i$. If $i \in S \backslash R$ then $\{i\} \cup R \subset T \in C$ and therefore $R \in \mathscr{S}_i$ and by the definition of L_i it follows that $m(R) \in L_i$. ∎

8. PROOF OF THE MAIN THEOREM

We will prove here that if $t_1, ..., t_n$ are "sufficiently large" then $N(\mathscr{K}(t_1, ..., t_n)) \subset C$, and therefore $N(\mathscr{K}(t_1, ..., t_n)) = C$. The proof proceeds in two steps. First we show that if $S \notin C$ and $|S| > d$ then $L(S) = \varnothing$. Next we show that if $S \notin C$ and $|S| \leqslant d$ then $L(S) \cap K_0 = \varnothing$ and therefore $K(S) = \varnothing$.

We start with a more complete description of the sets L_i and $L(S)$. For $d \geqslant k \geqslant 0$ define

$$A_k = \left\{p \in \mathbb{R}^d : p = \sum_{j=d-k}^{d-1} a_j x^j\right\}.$$

For $T \in [n]^{(k)}$, $k \leqslant d$ define

$$L^*(T) = L(T) \cap A_k.$$

LEMMA 8.1. (i) *For all* i, $L_i \subset M_i^+$.

(ii) $L^*(T)$ *is a closed parallelotope in* $\mathbb{R}^d$ *and* $L(T) = L^*(T) + M^0(T)$.

Proof of (i). If $S \in [i+1, n]^{(d)}$ then

$$H_i(S) = m(S)(-t_i) + (-t_i)^d = \prod_{\nu \in S} (t_\nu - t_i) > 0$$

therefore, $m(S) \in M_i^+$. (Remember: $0 < t_1 < \cdots < t_n$.) Define

$$W_i = \max(\{0\} \cup \{H_i(S)\colon S \in \mathscr{S}_i\}).$$

Since $H_i(S) > 0$ for all $S \in \mathscr{S}_i$ we obtain

$$L_i = \{p \in \mathbb{R}^d\colon 0 \leqslant H_i(p) \leqslant W_i\}. \quad \blacksquare$$

Proof of (ii). It is easy to verify that $\mathbb{R}^d = A_k + M^0(T)$. $L^*(T)$ is the set of all polynomials $p \in A_k$ satisfying

$$0 \leqslant p(-t_\nu) + (-t_\nu)^d \leqslant W_\nu \qquad (\nu \in T). \tag{8.1}$$

Since $A_k \cap M^0(T) = \{0\}$, (8.1) defines a closed parallelotope in $\mathbb{R}^d$. The fact that $L(T) = L^*(T) + M^0(T)$ is immediate. $\blacksquare$

Note that $m(T) \in A_k$ and satisfies $m(T)(-t_\nu) + (-t_\nu)^d = 0$ for all $\nu \in T$. Therefore, $m(T)$ is a vertex of $L^*(T)$. It is clear that if $|T| = d$ then $L^*(T) = L(T)$.

We consider now the families $\mathscr{K}$ and $\mathscr{L}$ for "sufficiently large" $(t_1, ..., t_n)$. We shall need some notations; for $T \in [n]^{(k)}$, $T = \{i_1 < i_2 < \cdots < i_k\}$, define

$$\pi(T) = \prod_{\nu \in T} t_\nu,$$

$$\psi(T) = \prod_{\nu=1}^{k} t_{i_\nu}^{\nu-1}.$$

LEMMA 8.2. *For* $(t_1, ..., t_n)$ *"sufficiently large,"*

(i) *If* $\mathscr{S}_i = \varnothing$ *then* $W_i = 0$. *If* $\mathscr{S}_i \neq \varnothing$ *then* $W_i = H_i(S_i^*)$, *where* S_i^* *is the last set in* $\mathscr{S}_i$ *w.r.t. the order* $<$.

(ii) $W_1 \geqslant W_2 \geqslant \cdots$.

Proof of (i). If $\mathscr{S}_i \neq \varnothing$ then $W_i = \max\{H_i(S)\colon S \in \mathscr{S}_i\}$ and $H_i(S) = \prod_{\nu \in S}(t_\nu - t_i)$. Thus the significant monomial in $H_i(S)$ is $\pi(S)$, and thus (Lemma 6.1) $H_i(S_1^*) \geqslant H_i(S)$ for all $S \in \mathscr{S}_i$.

Proof of (ii). We know (Lemma 5.4) that $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \cdots$ and therefore $S_1^* \geqslant S_2^* \geqslant \cdots$ and therefore, $W_1 \geqslant W_2 \geqslant \cdots$. (One can easily check that, in fact, $W_1 > W_2 > \cdots$.) ∎

We now need the following lemma:

LEMMA 8.3. *Let* $T \in [i, n]^{(k)}$ $(1 \leqslant i \leqslant n,\ k \leqslant d)$, *and suppose that* $\{i\} \cup T \notin C$. *Let q be a vertex of the parallelotope* $L^*(T)$, *let* $q - m(T) = \sum_{j=d-k}^{d-1} a_j x^j$. *Then there exists* $S \in [n]^{(\leqslant d)}$, $S < T$, *such that for "sufficiently large"* $(t_1, ..., t_n)$, $|a_j| = O(\pi(S))$ *for* $d-k \leqslant j \leqslant d-1$.

Remark. We shall use Lemma 8.3 in the following two cases:

A. $T \notin \mathcal{S}_i$ (i.e., $T \in [i+1, n]^{(d)}$ and $\{i\} \cup T \notin C$).

B. $T \in [n]^{(\leqslant d)}$ and $T \notin C$.

Proof. Let $T = \{i_1 < i_2 < \cdots < i_k\}$. A vertex q of $L^*(T)$ is a solution to the following system of equations:

$$q(-t_{i_\nu}) = -(-t_{i_\nu})^d + A_{i_\nu} \qquad \nu = 1, 2, ..., k,$$

where $A_{i_\nu} = 0$ or $A_{i_\nu} = W_{i_\nu}$. $m(T)$ is the solution to this system of equations where $A_{i_\nu} = 0$ for all $1 \leqslant \nu \leqslant k$.

If $\mathcal{S}_i = \varnothing$ then $W_{i_\nu} = 0$ for all $1 \leqslant \nu \leqslant k$, i.e., $L^*(T) = \{m(T)\}$. In this case, $a_i = 0$ for all i and Lemma 8.3 holds trivially for $S = \varnothing$.

Suppose $\mathcal{S}_i \neq \varnothing$. Put $S = \mathcal{S}_i^*$. Since $\{i\} \cup S \in C$, $\{i\} \cup T \notin C$, and $|S| \geqslant |T|$, it follows from Lemma 5.4 that $S < T$. Define

$$V = \begin{pmatrix} (-t_{i_1})^{d-k} & \cdots & (-t_{i_1})^{d-1} \\ \vdots & \cdots & \vdots \\ (-t_{i_k})^{d-k} & \cdots & (-t_{i_k})^{d-1} \end{pmatrix}$$

and

$$W = (A_{i_1}, ..., A_{i_k}).$$

For $0 \leqslant j \leqslant k-1$ denote by D_j the matrix obtained from V by replacing the $(j+1)$th column by W^{tr}.

Applying Cramer's formula for the system of equations for q and for $m(T)$ we get

$$a_{d-k+j} = \det D_j / \det V.$$

From Lemma 8.2 it follows that for every $1 \leqslant \nu \leqslant k$, $W_{i_\nu} = O(\pi(S))$ and therefore, $A_{i_\nu} = O(\pi(S))$.

We will now prove the assertion of Lemma 8.3, namely that $|a_j| = O(\pi(S))$ for all $d-1 \geqslant j \geqslant d-k$; in other words, we will show that

$$|\det D_j| = O(|(\det V)\,\pi(S)|). \tag{*}$$

Denote $\bar{T}=T\backslash\{i_1\}$. The significant monomial in the right side of (*) is

$$\pi(S)\prod_{\nu=1}^{k} t_{i_\nu}^{d-k+\nu-1}=\pi(S)\,\pi(T)^{d-k}\psi(T).$$

Evaluating $\det D_j$ by the $(j+1)$th column it is easy to see that for any minor X complement to an entry in the $(j+1)$th column we have

$$\det X=\pi(\bar{T})^{d-k}O\left(\prod_{\nu=2}^{j+1} t_{i_\nu}^{\nu-2}\prod_{\nu=j+2}^{k} t_{i_\nu}^{\nu-1}\right)=O(\pi(T)^{d-k}\psi(T)).$$

But $A_{i_\nu}=O(\pi(S))$ and we conclude that

$$\det D_j=O(\pi(S)\,\pi(T)^{d-k}\psi(T)). \quad \blacksquare$$

THEOREM 8.4. *Fr "sufficiently large" $(t_1,\ldots,t_n)$ the following assertion holds: If $R\subset[n]$, $|R|>d$, and $R\notin C$ then $L(R)=\varnothing$, i.e., $R\notin N(\mathscr{L})$.*

Proof. By Lemma 5.4 we may assume that $|R|=d+1$. Let $R=\{i\}\cup T$, where $T\in[i+1,n]^{(d)}$. Since $R\notin C$, $T\notin\mathscr{S}_i$. For "sufficiently large" $(t_1,\ldots,t_n)$ we will prove the following two assertions:

A. For every $S\in\mathscr{S}_i$

$$H_i(S)\ll H_i(T).$$

B. Let $q=q(t_1,\ldots,t_n)$ be a vertex of $L(T)$, then

$$|H_i(T)-H_i(q)|\ll H_i(T).$$

Assuming A and B the assertion $L(R)=L_i\cap L(T)=\varnothing$ follows easily. It is enough to show that for every $q\in L(T)$ and every $S\in\mathscr{S}_i$, $H_i(q)>H_i(S)$. It is enough to consider the case where q is a vertex of $L(T)$. From A it follows that for every $S\in\mathscr{S}_i$, $\frac{1}{2}H_i(T)>H_i(S)$. From B it follows that for every vertex q of $L(T)$, $H_i(q)>\frac{1}{2}H_i(T)$, therefore $H_i(q)>H_i(S)$. Now we prove claims A and B.

Proof of A. The significant monomial in $H_i(T)$ is $\pi(T)$, and the significant monomial in $H_i(S)$ is $\pi(S)$. From Lemma 5.4 it follows that $T>S$ and therefore, $H_i(T)\gg H_i(S)$.

Proof of B. Denote as before

$$q-m(T)=a_0+a_1x+\cdots+a_{d-1}x^{d-1}.$$

Then

$$H_i(q)-H_i(T)=a_0+a_1(-t_i)+\cdots+a_{d-1}(-t_i)^{d-1}.$$

From Lemma 8.3 it follows that there is $S \in [n]^{(\leqslant d)}$, $S < T$ such that

$$|H_i(q) - H_i(T)| \leqslant \sum_{j=0}^{d-1} |a_j|\, t_i^j = O(t_i^{d-1}\pi(S)).$$

But $H_i(T) = O(\pi(T))$ and since $T \in [i+1, n]^{(d)}$ and $T > S$, $\max(T \setminus S) \in T$ and $\max(T \setminus S) > i$, therefore $t_i^{d-1}\pi(S) = o(\pi(T))$. ∎

It remains to consider sets $T \notin C$ whose size is less than or equal to d. Let $T \in [n]^{(k)}$, $k \leqslant d$,

$$E(T) = M(T) + A_{k-1}.$$

Since $A_{k-1} \cap M^0(T) = \{0\}$, $E(T)$ is a hyperplane in $\mathbb{R}^d$. If $T \in [n]^{(\leqslant d)}$ and $T \notin C$ we will prove that for "sufficiently large" $(t_1, ..., t_n)$, K_0 can be separated from $L(T)$ by a hyperplane parallel to $E(T)$. We need first the following explicit description of $E(T)$:

LEMMA 8.5. *Let* $T = \{i_1 < i_2 < \cdots < i_k\}$ *then*

$$E(T) = \{p \in \mathbb{R}^d \colon \langle p, b \rangle = c\},$$

where $b = b(T) = b_0 + b_1 x + \cdots + b_{d-k} x^{d-k}$,

$$b_j = (-1)^{k+d+j} \sum_{l=1}^{k} (-1)^{l-1} t_{i_l}^{k-d+j-1} \Delta_{k-1}(t_{i_1}, ..., \hat{t}_{i_l}, ..., t_{i_k}).$$

($\hat{}$ *means deletion*) *for* $d-k \geqslant j \geqslant 0$ *and*

$$c = c(T) = \Delta_k(t_{i_1}, ..., t_{i_k}).$$

The proof of Lemma 8.5 is direct and is ommited.

It is clear that

$$M^0(T) + A_{k-1} = \{p \in \mathbb{R}^d \colon \langle p, b \rangle = 0\}.$$

THEOREM 8.6. *Let* $T \in [n]^{(\leqslant d)}$, $T \notin C$, *then for "sufficiently large"* $(t_1, ..., t_n)$, $L(T) \cap K_0 = \varnothing$, *i.e.*, $T \notin N(\mathscr{K})$.

Proof. Let $|T| = k$ and let $T = \{i_1 < i_2 < \cdots < i_k\}$. Define

$$u(T) = \min\{\langle p, b(T) \rangle \colon p \in L(T)\}.$$

Recall that $L(T) = L^*(T) + M^0(T)$ (Lemma 8.1). For $p \in M^0(T)$, $\langle p, b(T) \rangle = 0$, and therefore

$$u(T) = \min\{\langle p, b(T) \rangle \colon p \in L^*(T)\}.$$

Since $L^*(T)$ is closed, $u(T)$ exists, define

$$A(T)=\{p\in\mathbb{R}^d\colon \langle p, b(T)\rangle < u(T)\}.$$

It is clear that $A(T)\cap L(T)=\varnothing$.

We show that for "sufficiently large" $(t_1,..., t_n)$, $K_0\subset A(T)$, and this implies that $K_0\cap L(T)=\varnothing$. It is enough to show that for "sufficiently large" $(t_1,..., t_n)$, for every $S\in \operatorname{skel}_{d-1} C$, $m(S)\in A(T)$. Exactly as in the proof of Theorem 8.4, it suffices to prove the following two assertions:

A. If $q=q(t_1,..., t_n)$ is a vertex of $L^*(T)$ then

$$c(T)\gg |c(T)-\langle q, b(T)\rangle|.$$

B. If $S\in \operatorname{skel}_{d-1} C$ then

$$c(T)\gg \langle m(S), b(T)\rangle.$$

We shall need the following facts:

$$c(T)=\prod_{\nu=1}^{k} t_{i_\nu}^{\nu-1}+o=\psi(T)+o.$$

For $d-k\geqslant j\geqslant 1$,

$$|b_j|=\left(\prod_{\nu=2}^{k} t_{i_\nu}^{\nu-2}\right)t_{i_1}^{k-d+j-1}+o=\left(\prod_{\nu=1}^{k} t_{i_r}^{\nu-2}\right)t_{i_\nu}^{k-d+j}+o$$

(for $j>d-k$, $b_j=0$).

Proof of A. Put $q-m(T)=\sum_{j=d-k}^{d-1} a_j x^j$; note that

$$|\langle q-m(T), b(T)\rangle|=|a_{d-k}b_{d-k}|.$$

From Lemma 8.3 it follows that for every j, $a_j=o(\pi(T))$ and therefore

$$|a_{d-k}b_{d-k}|=o\left(\prod_{\nu=1}^{k} t_{i_\nu}^{\nu-2}\pi(T)\right)=o(\psi(T)).$$

This gives

$$c(T)=\psi(T)+o\gg |\langle q-m(T), b(T)\rangle|.$$

Proof of B. Put $|S|=l$ and

$$m(S)=\left(x^{d-l}\prod_{\nu\in S}(x+t_\nu)\right)-x^d=\sum_{j=d-l}^{d-1} z_j x^j.$$

Denote, as before, $b(T)=\sum_{j=0}^{d-k} b_j x^j$. If $k>l$ then $d-l>d-k$ and

$$\langle m(S), b(T)\rangle = 0 \ll c(T).$$

Let $k \leqslant l$, from Lemma 5.4 it follows that $T>S$ and therefore $\pi(T)>\pi(S)$. Remember that

$$c(T)=\psi(T)+o=\prod_{v=1}^{k} t_{i_v}^{v-2}\pi(T)+o.$$

We will show that for all $d>j\geqslant 0$,

$$|z_j b_j| \ll c(T).$$

In fact,

$$|b_j| = \prod_{v=1}^{k} t_{i_v}^{v-2} t_{i_1}^{k-d+j}+o, \qquad (0 \leqslant j \leqslant d-k),$$

and

$$z_j = O(\pi(S)) \qquad (d-l \leqslant j \leqslant d-1),$$

therefore

$$|z_j b_j| = O\left(\prod_{v=1}^{k} t_{i_v}^{v-2}\, t_{i_1}^{k-d+j}\pi(S)\right)$$
$$\ll \prod_{v=1}^{k} t_{i_v}^{v-2} t_{i_1}^{k-d+j}\pi(T) \leqslant \prod_{v=1}^{k} t_{i_v}^{v-2}\pi(T) = \psi(T),$$

i.e., $|z_j b_j| \ll c(T)$. This completes the proof of Theorem 8.6. ∎

Theorems 8.6 and 8.4 complete the proof of the sufficiently of Eckhoff's conditions.

Remark. If $f_{d-1}=\binom{n}{d}$ then already $N(\mathscr{L})=C$, and thus f is the f-vector of a family of "strips" in $\mathbb{R}^d$.

9. Some Remarks and Open Problems Concerning Families of Simplicial Complexes Discussed in this Paper

An implication diagram for properties of simplicial complexes discussed in this paper is given in Fig. 1.

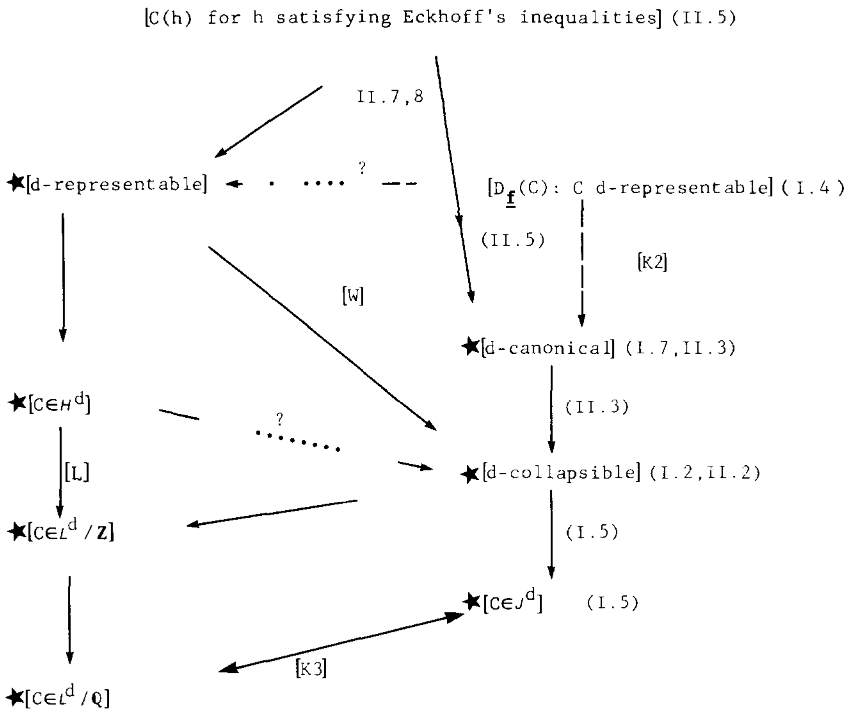

FIG. 1. The implication shown by the dashed line holds for a basis f_0 in general position w.r.t. the standard basis.

Some explanations are in order: The families of simplicial complexes $\mathcal{L}^d/\mathbb{Z}$, $\mathcal{L}^d/\mathbb{Q}$, $\mathcal{H}^d$, and $\mathcal{J}^d$ are defined as follows: $C \in \mathcal{L}^d/\mathbb{Z}$ if

$$H_i(C') = 0 \qquad \text{for all } i \geqslant d \text{ and all } C' = C/S,\ S \in C. \tag{9.1}$$

$C \in \mathcal{L}^d/\mathbb{Q}$ if (9.1) holds for homology with rational coefficients. $C \in \mathcal{H}^d$ if $C = N(\mathcal{K})$, where $\mathcal{K}$ is a family of homology cells in $\mathbb{R}^d$ such that every non-empty intersection of members of $\mathcal{K}$ is again a homology cell. $C \in \mathcal{J}^d$ if $H^{k+1}_{d+k}(C) = 0$ for all $k \geqslant 0$. (The generalized homology groups $H^p_k(C)$ were defined in Part I, Sect. 5.)

The properties marked by ★ in Fig. 1 remain true under the following two operations:

(1) taking quotient complexes;

(2) taking induced subcomplex on a subset of the vertices.

The equivalence $C \in \mathcal{L}^d/\mathbb{Q} \Leftrightarrow C \in \mathcal{J}^d$ is proved in [K3]. This result and Theorem 6.1 in Part I imply that f-vectors of complexes in any of the classes $\mathcal{L}^d/\mathbb{Q}$, $\mathcal{L}^d/\mathbb{Z}$, and $\mathcal{H}^d$ are characterized by inequalities (1.5).

Let C be a simplicial complex on the vertex set $[n]$ and let V be a real n-dimensional vector space with a fixed basis $\mathbf{e} = (e_1, \ldots, e_n)$. In Part I, Section 4 we defined for each basis $\mathbf{f} = (f_1, \ldots, f_n)$ of V, a simplicial complex $D_{\mathbf{f}}(C)$. The following result can be obtained by similar methods to those in Part I, Section 6: If (and only if) $C \in \mathscr{J}^d$ and $\mathbf{f}$ is a basis in general position w.r.t. $\mathbf{e}$ (see I. Sect. 3) then $D_{\mathbf{f}}(C)$ is d-canonical. (See [K2]; this result together with some basic facts on d-canonical complexes give an alternative proof for Theorem 6.1 in Part I.)

Generically, the complex $D_{\mathbf{f}}(C)$ does not depend on $\mathbf{f}$. Moreover, denote by $D(C)$ the simplicial complex obtained from C in the generic case, then $D(C)$ is an initial complex w.r.t. the partial order $\lhd$, i.e., if $T \in D(C)$, $|S| = |T|$, and $S \lhd T$ then $S \in D(C)$ (see [K2]). Call a simplicial complex C *initial* if it is initial w.r.t. the partial order $\lhd$.

It is easy to see that if C is initial then $C \in \mathscr{J}^d$ iff C is d-canonical and thus the result stated above can be reformulated (in somewhat less generality) as: $C \in \mathscr{J}^d \Leftrightarrow D(C) \in \mathscr{J}^d$. For d-representable complexes perhaps more is true:

PROBLEM 9.1. Let C be a d-representable complex. Must $D(C)$ be d-representable? (Perhaps it is even always true that $D_{\mathbf{f}}(C)$ is d-reresentable for every basis $\mathbf{f}$.)

For $d = 1, 2$ the answer for Problem 9.1 is positive by the following result:

PROPOSITION 9.2. *If $d \leqslant 2$ and C is an initial d-canonical complex on the vertex set $[n]$ then C is d-representable.*

The proof of Proposition 9.2 is by a construction similar to, but much simpler than, the construction given in Section 7. It is unlikely that the assertion of Proposition 9.2 is true for $d > 2$.

It would be interesting to characterize f-vectors of complexes represented as nerves of families of sets taken from a restricted class of convex sets. For example,

PROBLEM 9.3. Characterize f-vectors of nerves of families of *affine spaces* in $\mathbb{R}^d$.

Finally, we would like to remark that the family of d-representable complexes is far from being understood; for example, very little is known about the following problem:

PROBLEM 9.4. Which graphs (i.e., 1-dimensional complexes) are 2-representable?

10. Eckhoff's Conjecture and McMullen's g-Conjecture

We already remarked in Part I, Section 7.2, that Eckhoff's conjecture resembles McMullen's g-conjecture concerning f-vectors of simplicial polytopes (see [McM, MS]). McMullen's conjecture was solved by Billera and Lee [BL] (sufficiency) and Stanley [S] (necessity). In fact, the constructions given in this paper have some similar points with the constructions of Billera and Lee proving the sufficiency of McMullen's conjecture. In both cases the starting point is a cyclic d-polytope (implicitly in our construction). Even more significantly, in both cases the initial construction depends on n variables $t_1,..., t_n$, and the existence of appropriate real numbers $\bar{t}_1,..., \bar{t}_n$ for which the construction will satisfy the required properties is done by appropriate calculations in the ordered field of rational functions in n variables over $\mathbb{R}$.

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